Perturbation Theory with Convergent Series: The Calculation of the $\lambda\varphi^4_{(4)}$- field theory β-function

The problem of series summation

A typical problem: Let $F(\lambda)$ be given by a formal power series

$$F(\lambda) \sim \sum_{k=0}^{\infty} f_k \, \lambda^k \ .$$

Let $\{f_k\}_{k=1,\ldots,N}$ be known.

How to calculate the quantity $F(\lambda)$
(especially, for $\lambda \gg 1$)?

- The series is asymptotic
- The quantity $F(\lambda)$ is non-analytic at the origin

The asymptotics of the f_k $(k \to \infty)$ is known:

$$\tilde{f}_k := C a^k \, k^b \, \Gamma(k+d) \left\{ A_0 + \frac{A_1}{k} + O\left(\frac{1}{k^2}\right) \right\} \ .$$

The Borel summation:

$$F(\lambda) = \int_0^{\infty} e^{-t} \, B(t) \, dt \ , \quad B(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f_k$$

- Only *finite* number of f_k's is known;
- Exact analytical properties of $F(\lambda)$ are hard to obtain;

Perturbation Theory with Convergent Series (PTCS)

- The method was proposed by V. V. Belokurov, Yu. P. Solovyov and E. T. Shavgulidze (Moscow State University)

Consider "the 0-dimensional analogue" of path integral

$$I(\lambda) = \int_0^{\infty} e^{-x^2 - \lambda\, x^{2m}}\, dx = \exp\left(\frac{1}{8\lambda}\right)\, K_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right)\frac{1}{\sqrt{4\lambda}}\ ,$$

or

$$I(\lambda) = \sum_{k=0}^{\infty} f_k \lambda^k\ , \quad f_k = \frac{(-1)^k}{k!}\frac{\sqrt{\pi}}{2^{4k}}\frac{(4k)!}{(2k)!}\ .$$

Approximate $I(\lambda)$ with some new integral $J(\lambda, R)$

$$|I(\lambda) - J(\lambda, R)| \leq \varepsilon\ ,$$

The series for $J(\lambda, R)$ is *convergent*

$$J(\lambda, R) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\lambda^{\frac{2k}{4}}\, A_{2k}(R)\, B_{\frac{k}{2}}$$

$$= \sum_{k \text{ even}}^{\infty} \frac{1}{(2k)!}\lambda^{\frac{k}{2}}\, A_{2k}(R)\, B_{\frac{k}{2}} + \sum_{k \text{ odd}}^{\infty} \frac{1}{(2k)!}\lambda^{\frac{k}{2}}\, A_{2k}(R)\, B_{\frac{k}{2}}$$

$$\xrightarrow{R\to\infty} \sum_{k}^{\infty} \lambda^k\, \frac{(-1)^k}{k!}\, \Gamma\left(\frac{4k+1}{2}\right)\ ,$$

where

$$B_{\frac{k}{2}} = \Gamma\left(\frac{2k+1}{2}\right)$$

$$A_k(R) = \frac{i^k}{2\pi}\int_{-R}^{+R} \rho^k \int_{-\infty}^{+\infty} e^{-r^4} e^{-i\rho r}\, dr\, d\rho$$

- With PTCS, $I(\lambda)$ can be calculated precisely, although only *finitely many* of its PT coefficients are known.

PTCS as a method for divergent series summation

Consider a series

$$\sum_k f_k \lambda^k$$

Assume

$$|f_n| \sim C\, a^n\, n^b\, n! \ .$$

Then, there exists a function $f(t) \in C^\infty([0,\infty))$ s. t.

$$f_n = \frac{f^{(n)}(0)}{n!} \ .$$

Let the function $f(t)$ satisfy

- $(-1)^k f^{(k)}(t) \geq 0 \ \forall k \geq 0, t \in [0,\infty)$;
- allows analytic continuation in $\mathrm{Re}\,\lambda > 0$;
- $\left| f(\lambda) - \sum_{n=0}^{N-1} f_n \lambda^n \right| < C_1^N\, a^N\, N^b\, N!$ holds uniformly in λ, $\mathrm{Re}\,\lambda > 0$;

Then,

$$f(\lambda) = \int_0^\infty e^{-\lambda t}\, \mu(dt) \ ,$$

where $\mu(dt)$ is a positive Radon measure on $[0,\infty)$.

$$f(\lambda) \approx \sum_{n=0}^{N} \frac{1}{(2n)!}\, \lambda^{\frac{n}{m}}\, A_{2n}(m,R)\, B_{\frac{n}{m}} \ ,$$

$$B_{\frac{n}{m}} = \int_0^\infty x^{\frac{n}{m}}\, \mu(dt) \ ,$$

$$A_{2n}(m,R) = (-1)^n \int_{-R}^{+R} \tilde{\varphi}_{2m}(\rho)\, \rho^{2n}\, d\rho \ ,$$

$$\tilde{\varphi}_{2m}(\rho) := \frac{1}{2\pi} \int e^{-i\rho r} e^{-r^{2m}}\, dr \ .$$

The β-function

Consider $O(n)$-symmetric field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a + \frac{1}{2}m_B^2\varphi^a\varphi^a + \frac{16\pi^2}{4!}\lambda_B(\varphi^a\varphi^a)^2\,, \quad a = 1,\ldots,n$$

The β-*function* is (in the $\overline{MS}$-scheme)

$$\beta(\lambda) = \sum_{k=2}^{\infty}\beta_k\lambda^k = 1.66\lambda^2 - 3.33\lambda^3 + 19.97\lambda^4 - 175.25\lambda^5 + 1898.85\lambda^6 + \ldots\,.$$

$$f(\lambda) = \exp(-\beta(\lambda)/\lambda) = 1 - 1.6\lambda + 4.7\lambda^2 - 26.3\lambda^3 + 219\lambda^4 - 2297\lambda^5\,.$$

The asymptotics for the β_k is known; for $n = 2$

$$\beta_k \sim \frac{0.543}{16\pi^2}\,k^{\frac{7}{2}}\,k!\left\{1 - \frac{4.82}{k} + O\left(\frac{1}{k^2}\right)\right\}\,.$$

The coefficients A ($R = 6$)

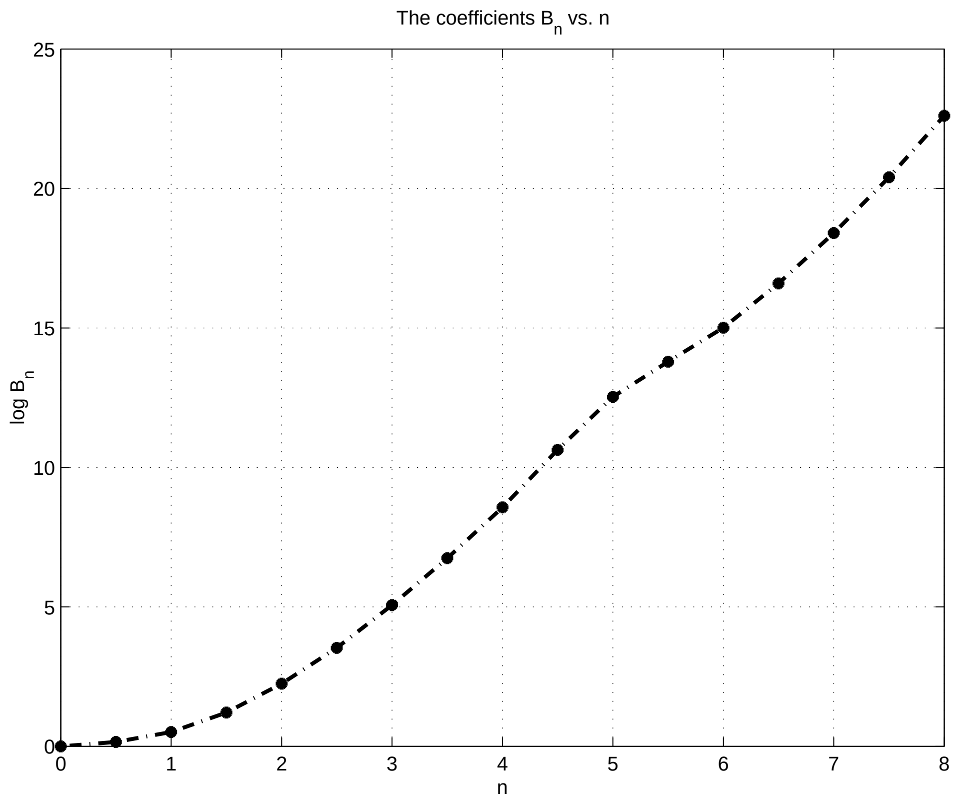
n	$\lvert A_{2n}(2,6)\rvert$	n	$\lvert A_{2n}(2,6)\rvert$	n	$\lvert A_{2n}(2,6)\rvert$
1	0.995345	4	1526.6842	7	$3.5515064\cdot10^7$
2	0.415867	5	42593.555	8	$1.0668836613\cdot10^9$
3	51.892553	6	1212765.477	9	$3.2743551347\cdot10^{10}$

The coefficients $B_{n/2}$

n	$B_{n/2}$	n	$B_{n/2}$
1	1.16971	9	41374.43266
3	3.34822	11	976968.175
5	34.16947	13	$1.6115094688\cdot10^7$
7	845.48256	15	$7.21784871286\cdot10^8$

The β-function values

λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
R	8.4	7.5	6.5	6.6	6.6	6.6	6.4	6.5	6.4	6.5
β	0.01	0.05	0.12	0.20	0.32	0.62	0.87	1.2	1.5	1.9
ε	0.01	0.01	0.03	0.05	0.08	0.1	0.4	0.4	0.6	0.8


The coefficients B_n vs. n
log B_n
n
0
5
10
15
20
25
0
1
2
3
4
5
6
7
8