A new Monte Carlo method of the numerical integration

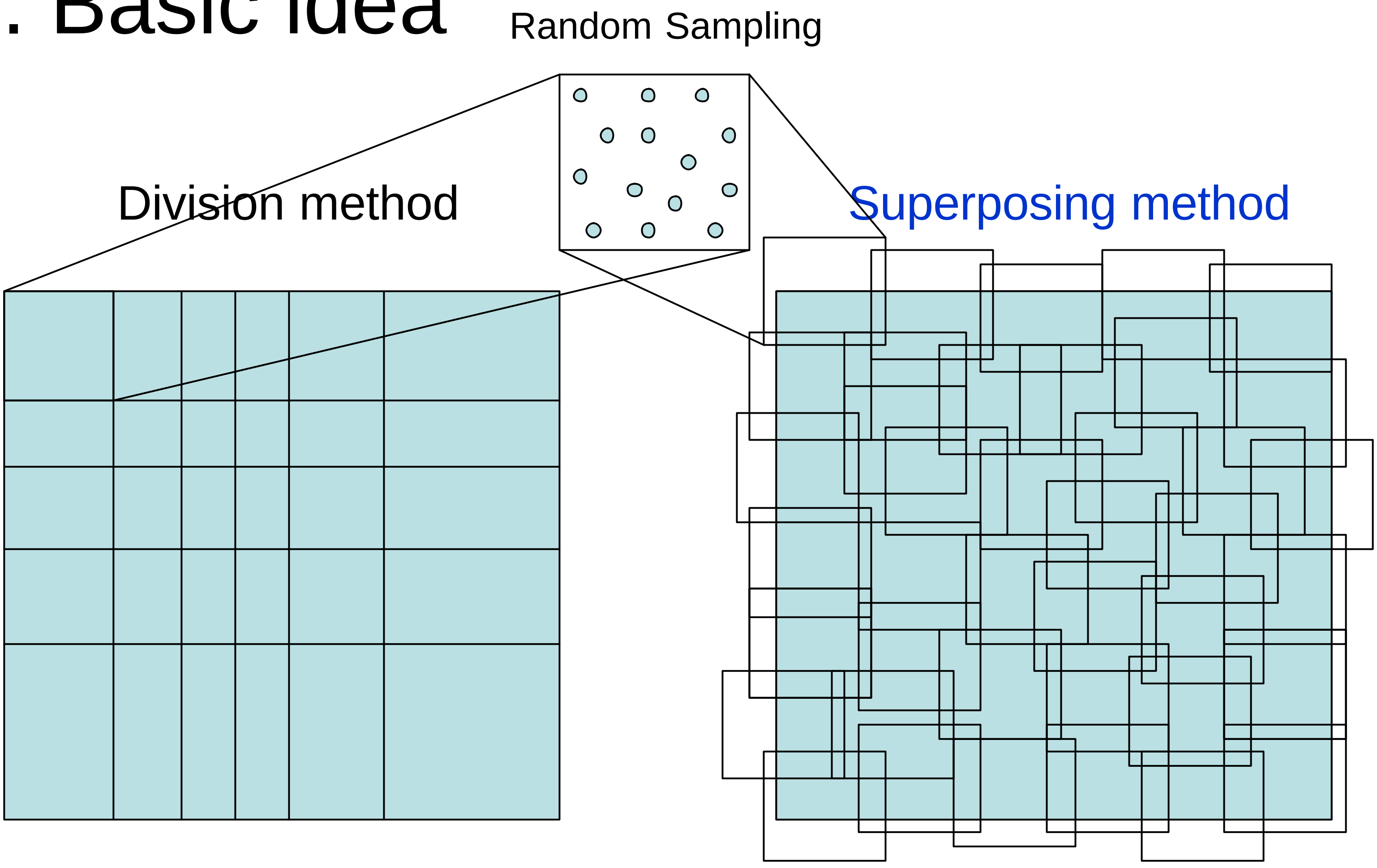
"Superposing Method"

T.Kaneko and K.Tobimatsu*

KEK, Tsukuba Japan

*Kogakuin University, Tokyo Japan

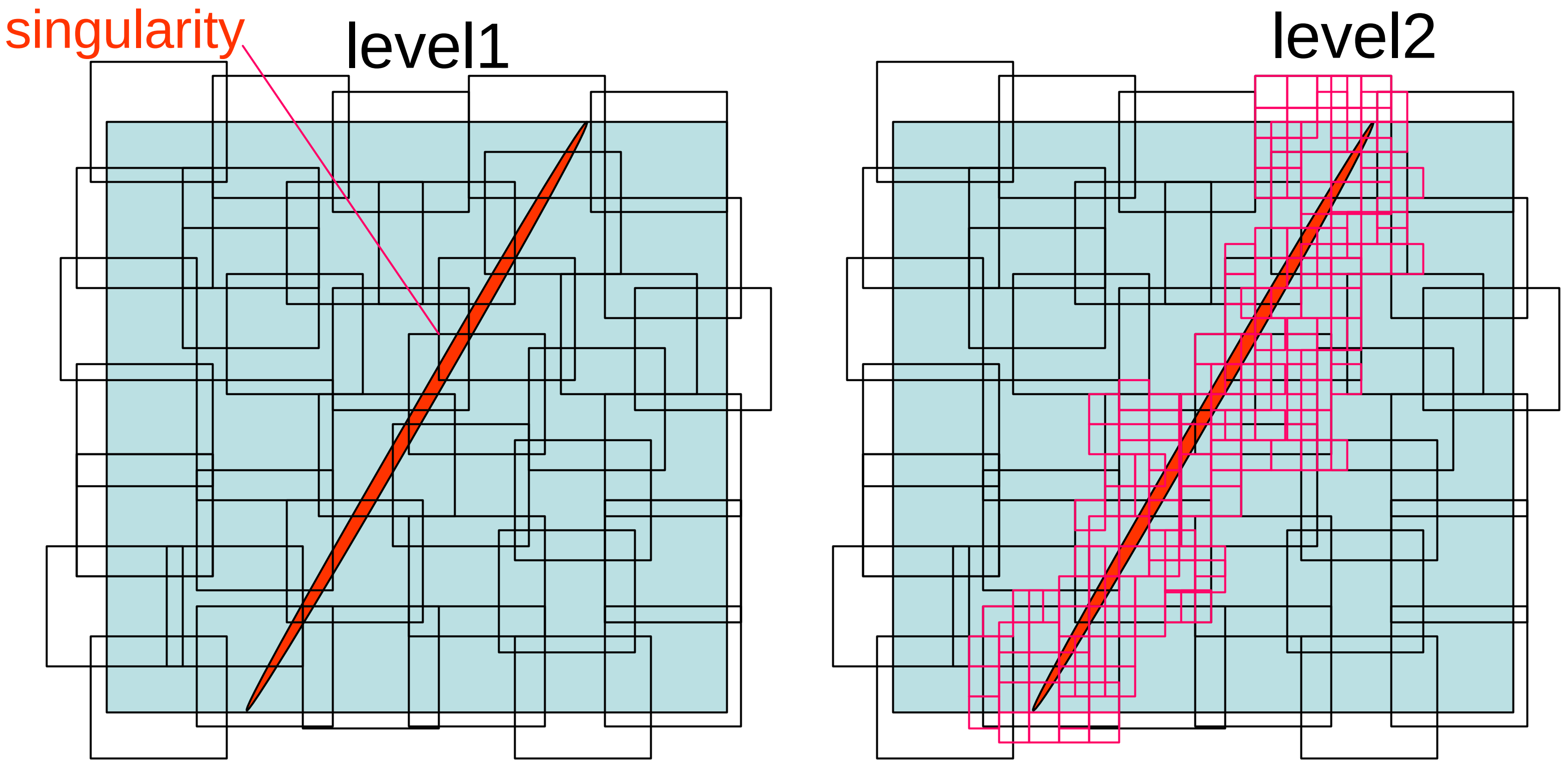
1. Basic idea



Regions are deterministic

Regions are randomly taken

The method is more flexible than usual division method in choosing a shape for sub-region. The shape can be changed for each level.



2. General Formulation

Naïve MonteCarlo

S : Integration volume

N:Number of samplings

$$I = \int_S f(x)dx = \frac{S}{N}\sum_{i=1}^{N} f(x_i)$$

We extend the above expression.

1. Formally we enlarge the integration volume beyond S.
2. Introduce a probability distribution as a weight , $t(x), x \in R^n$

$$\int d^n x t(x) = 1$$

For example, we consider a region r which include the origin of coordinate axes inside and whose volume is T. $t(x)$ takes a constant value $1/T$ in the region and takes a zero value outside.

$$t(x) = \begin{cases} 1/T & x \in r \\ 0 & otherwise \end{cases}$$

We multiply "1" on the integral I

$$I = \int d^n x f(x)$$

$$= \int d^n x f(x) \int d^n z t(z)$$

Changing the variables *(x,z)* into *(y,x)* where *z=y-x*, we have

Master Formula : $$I = \int d^n y \int d^n x f(x) t(y-x)$$

We apply Monte Calro method to the last expression :

1. A y value is randomly chosen then $t(y\text{-}x)$ is a function of x.
2. The meaning of x integration is to compute the average of $f(x)$ around y using the probability distribution $t(y\text{-}x)$.
3. We take many sampling for y. For each y, we have average value of $f(x)$ by the distribution $t(y\text{-}x)$ and finally we obtain the average of each estimate.

Examples of t(y)

(1-dimension)

A range of Δ

$$t(y)=\frac{1}{\Delta}q(\frac{\Delta}{2}-|y|)$$

(2-dimension)

A rectangle

$$t(y_1,y_2)=\frac{1}{\Delta_1\Delta_2}q(\frac{\Delta_1}{2}-|y_1|)q(\frac{\Delta_2}{2}-|y_2|)$$

A circle

$$t(y_1,y_2)=\frac{4}{p\Delta^2}q(\frac{\Delta}{2}-\sqrt{y_1^2+y_2^2})$$

An ellipse and the other shapes are , in principle, OK.

3. An example of 1-dim. integration

$$I = \int_{-\infty}^{+\infty} dx\, f(x) = \int_{-\infty}^{+\infty} dy\, J_{\Delta}(y)$$

$J_{\Delta}(y)$ is the average of $f(x)$ over the range of Δ around y,

$$J_{\Delta}(y) \equiv \frac{1}{\Delta} \int_{y-\frac{\Delta}{2}}^{y+\frac{\Delta}{2}} dx f(x)$$

When the integration range is finite, i.e., $I = \int_a^b dx f(x)$, we introduce

step functions in the integrand: $f(x) \rightarrow f(x) q(b-x) q(x-a)$

$$I = (b - a + \Delta) J$$

$$J = \frac{1}{b - a + \Delta} \int_{a-\frac{\Delta}{2}}^{b+\frac{\Delta}{2}} dy\, J_{\Delta}(y)$$

$$I = (b - a + \Delta) J$$

$$J = \frac{1}{b - a + \Delta} \int_{a-\frac{\Delta}{2}}^{b+\frac{\Delta}{2}} dy\, J_{\Delta}(y)$$

$$\Delta J_{\Delta}(y) \equiv \int_{a'}^{b'} dx f(x), \quad a' = y - \frac{\Delta}{2}, \quad b' = y + \frac{\Delta}{2}$$

Recursive Method

$$\Delta J_{\Delta}(y) = (b' - a' + \Delta') J'$$

$$J' = \frac{1}{b' - a' + \Delta'} \int_{a'-\frac{\Delta'}{2}}^{b'+\frac{\Delta'}{2}} dz\, J'_{\Delta'}(z)$$

$$J'_{\Delta'}(z) \equiv \frac{1}{\Delta'} \int_{z-\frac{\Delta'}{2}}^{z+\frac{\Delta'}{2}} dx f(x)$$

We introduce g(y)>0, which is a probability distribution function.

$$J = \frac{1}{b-a+\Delta}\int_{a-\frac{\Delta}{2}}^{b+\frac{\Delta}{2}} dy\, g(y) \frac{1}{\Delta}\int_{y-\frac{\Delta}{2}}^{y+\frac{\Delta}{2}} dx \frac{f(x)}{g(y)}$$

$$\frac{1}{b-a+\Delta}\int_{a-\frac{\Delta}{2}}^{b+\frac{\Delta}{2}} dy\, g(y) = 1$$

$$\frac{1}{M}\sum_{j=1}^{M} g(y_j) = 1 \qquad g(y_j) \equiv \frac{MN_j}{N}$$

$$J = \frac{1}{M}\sum_{j=1}^{M} g(y_j) J_j$$

$$J_j = \frac{1}{N_j}\sum_{k=1}^{N_j} \frac{f(x_k)}{g(y_j)} \qquad \sum_{j=1}^{M} N_j = N$$

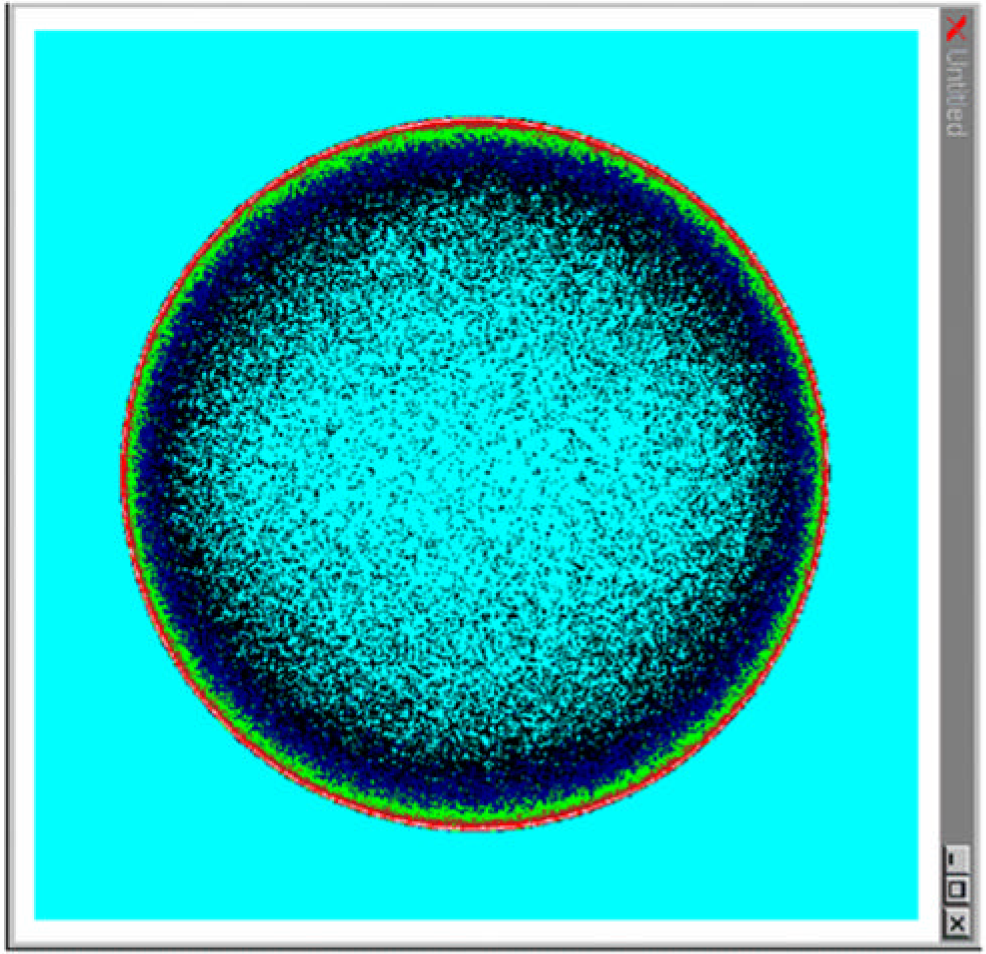
M:Number of samplings for dy integration

N_j:Number of samplings for dx integration around y_j

$$I_3 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dx_1 dx_2 dx_3 \frac{erq\,(1-r^2)}{(r^2-a^2)^2+e^2}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$e = 10^{-5} \qquad a = 0.8$$



High Pink

Density

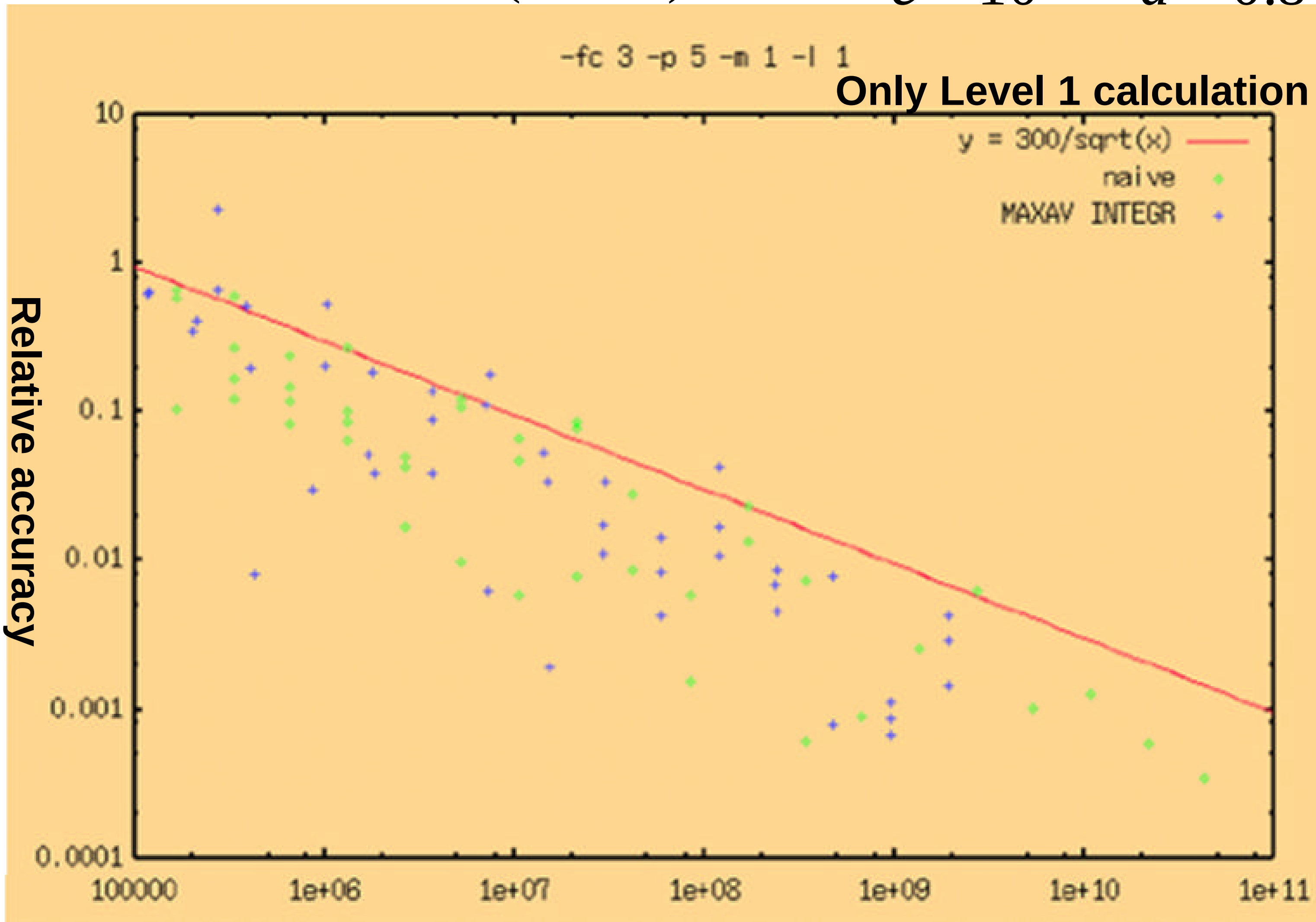
Low Lightblue

Sampling density distribution

$$I_3 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} dx_1 dx_2 dx_3 \, \frac{erq\,(1-r^2)}{(r^2-a^2)^2+e^2}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$e = 10^{-5} \qquad a = 0.8$$



4. Use of the singularity information

We assume that a ridge of the integrand is parameterized by a function $t(s),\ s \in R^n$.

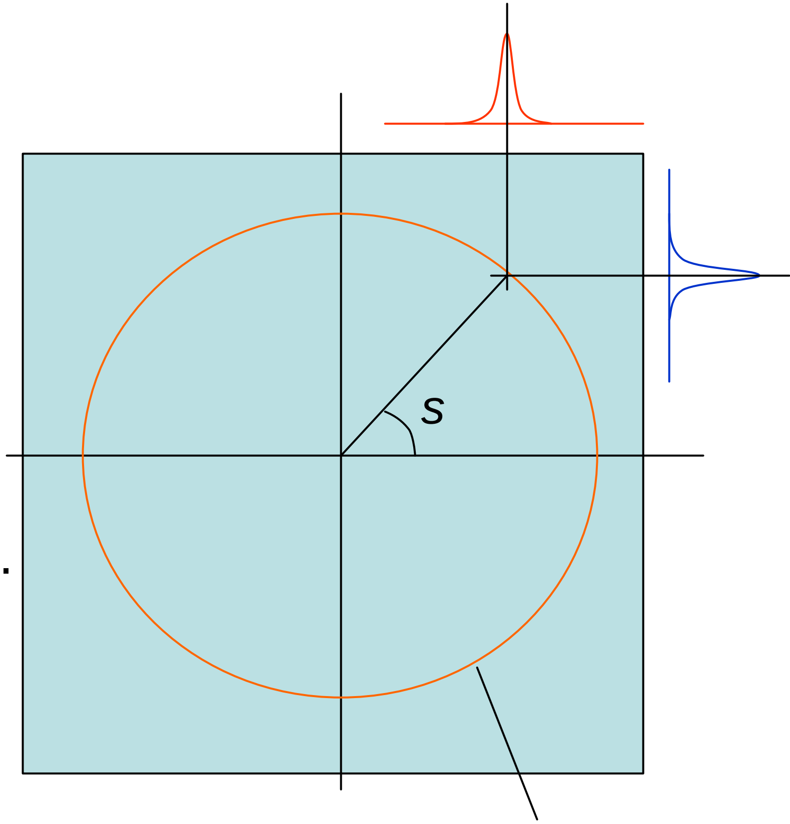
$w(x)$ is a probability distribution function.

$$I = \int d^n x f(x)$$

$$= \int d^n x w(x-t) \frac{f(x)}{w(x-t)}$$

$$= \frac{1}{S} \int d^m s \int d^n x w(x-t(s)) \frac{f(x)}{w(x-t(s))}$$

Volume of parameter space



Circular Singularity

For example, ***w*** could be a Normal Distribution.

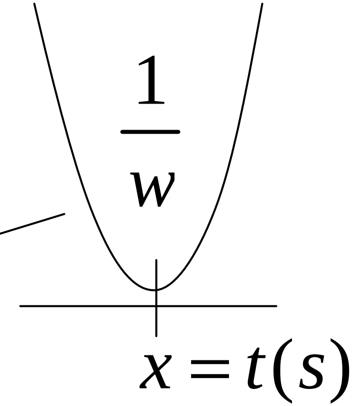
m is the freedom of the singularity.

$$w(x) = \prod_{k=1}^{n} \frac{e^{-\frac{x_k^2}{s_k^2}}}{\sqrt{2p\, s}\ _k}$$

We concentrate samplings around the singular region.

We use random numbers of the Normal distribution.

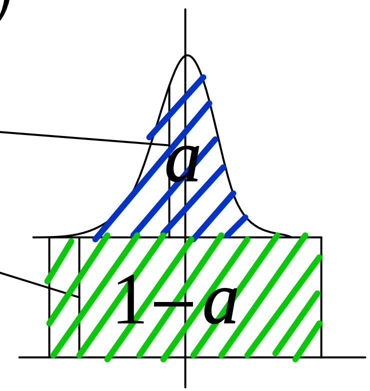
$$I = \frac{1}{S}\int d^m s \int \underline{d^n x w(x - t(s))} \frac{f(x)}{w(x - t(s))}$$



Denominator becomes very small far from x=t(s).
We add a uniform distribution w_0.

$$W(x - t(s)) = a w(x - t(s)) + (1 - a) w_0 (x - t(s))$$

w : normal distribution
w_0 : uniform distribution



$$I = \frac{1}{S}\int d^m s \int d^n x W(x - t(s)) \frac{f(x)}{W(x - t(s))}$$

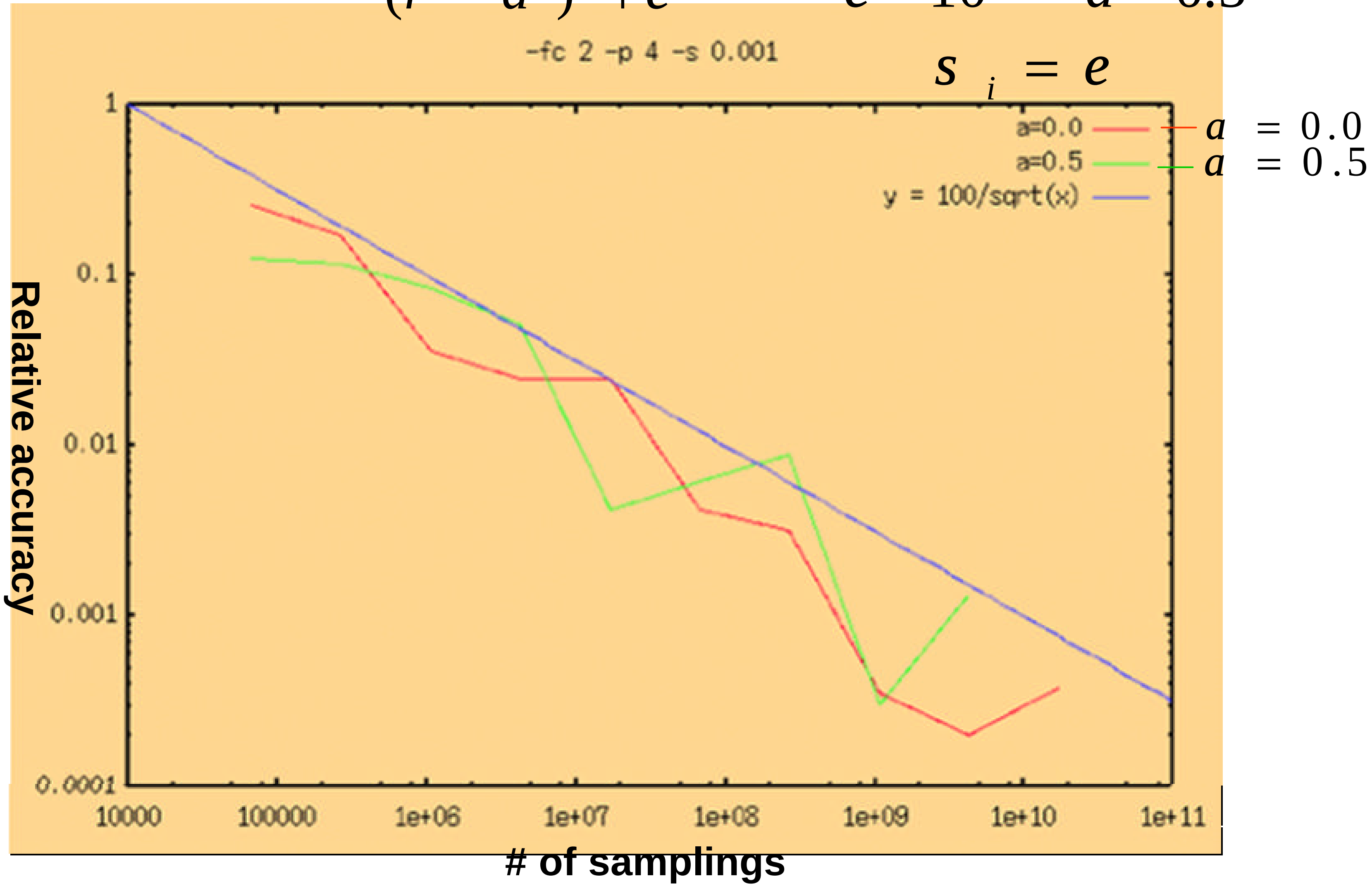
$$= \frac{1}{S}\int d^m s \left[a \int d^n x w \frac{f(x)}{W} + (1 - a) \int d^n x w_0 \frac{f(x)}{W} \right]$$

$$I_2 = \int_{-1}^{1}\int_{-1}^{1} dx_1 dx_2 \frac{e x_2^2 q (1-r^2)}{(r^2-a^2)^2+e^2}$$

$$r = \sqrt{x_1^2 + x_2^2}$$

$$e = 10^{-4} \quad a = 0.5$$

$$s_i = e$$



Summary

- We proposed a new Monte Carlo integration method in which sub-regions are randomly taken.
- We proposed a method of using the singularity information when the location can be parameterized.
- We are developing an integration program which uses the methods.