ACAT 2002

Computation of Cohomology of Lie (Super)Algebra: Algorithms, Implementation and New Results

Vladimir V. Kornyak

LIT, JINR, Dubna

kornyak@jinr.ru

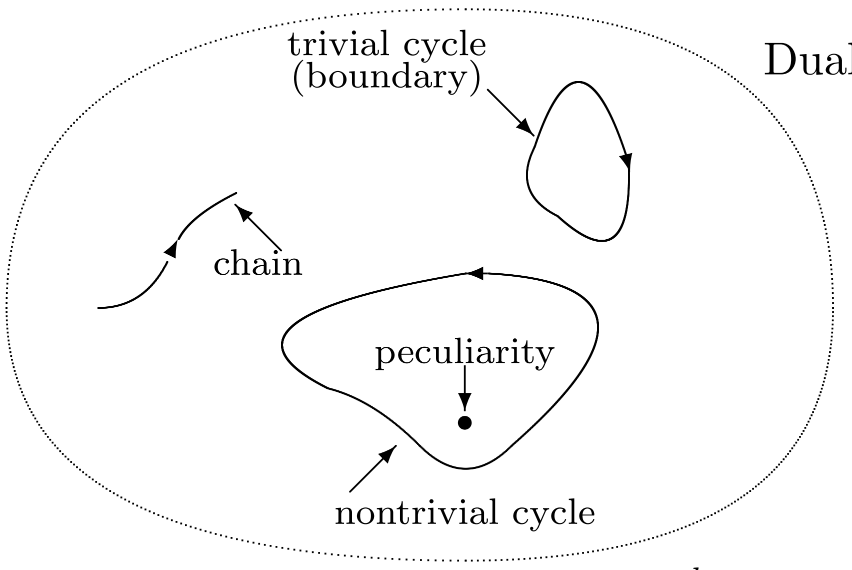
June 25, 2002

Homology and cohomology

Chain and cochain complexes:

$$0 \leftarrow C_0 \xleftarrow{\partial_0} \cdots \xleftarrow{\partial_{k-2}} C_{k-1} \xleftarrow{\partial_{k-1}} C_k \xleftarrow{\partial_k} C_{k+1} \xleftarrow{\partial_{k+1}} \cdots$$

$$0 \rightarrow C^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{k-2}} C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \cdots .$$



Duality $\int_M d\omega = \int_{\partial M} \omega$ – Stokes' theorem:

$\int_a^b f(x)dx = F(b) - F(a)$ – Newton–Leibniz;

$\int_S (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{a} = \oint_{\partial S} \overrightarrow{F} \cdot d\overrightarrow{s}$ – Stokes;

$\int_V (\nabla \cdot \overrightarrow{F}) \cdot dV = \int_{\partial V} \overrightarrow{F} \cdot d\overrightarrow{a}$ – Gauss–Ostrogradski.

Cocycles $Z^k = \mathrm{Ker}\ d^k = \{C^k \mid dC^k = 0\}$.

Coboundaries $B^k = \mathrm{Im}\ d^{k-1} = \{C^k \mid C^k = dC^{k-1}\}$.

Cohomology	$H^k = Z^k / B^k$

Cohomology of Lie (super)algebra A in the module X

$C^k = C^k(A; X)$ is a space of super skew-symmetric k-linear mappings $A \times \cdots \times A \to X$, ($C^0 = X$ by definition).

Differential:

$(d^k c)(a_0, \ldots, a_k)$

$$= \; - \sum_{0 \le i < j \le k} (-1)^{s(a_i)+s(a_j)+p(a_i)p(a_j)} c([a_i, a_j], a_o, \ldots, \widehat{a}_i, \ldots, \widehat{a}_j, \ldots, a_k)$$

$$- \sum_{0 \le i \le k} (-1)^{s(a_i)} a_i c(a_o, \ldots, \widehat{a}_i, \ldots, a_k).$$

$c(\ldots) \in C^k; \qquad a_i \in A; \qquad p(a_i) \in \mathbb{Z}_2$ – parity of a_i;

$$s(a_i) = \begin{cases} i, & p(a_i) = 0 \ (a_i \textit{ is even}) \\ \textit{number of even elements in } a_0, \ldots, a_{i-1}, & p(a_i) = 1 \ (a_i \textit{ is odd}) \end{cases}$$

Some interpretations in low dimensions (degrees) k

Trivial module

$H^1(A) \sim (A/[A,A])'$ describes "deviation of A from simplicity"

$H^2(A)$ describes nontrivial central extensions of A

Adjoint module

$H^1(A;A) \sim \mathrm{Der}A/\mathrm{ad}A$ describes "external derivations of A",
i.e., quotient space of all derivations w.r.t. internal derivations

$H^2(A;A)$ describes infinitesimal deformations of A

Main cause of computational difficulties

Basis of C^k for Lie superalgebra

$$C(e_{i_1}, \dots, e_{i_k}; a_\alpha) \equiv C(e_{i_1}) \wedge \dots \wedge C(e_{i_k}) \otimes a_\alpha \equiv e'_{i_1} \wedge \dots \wedge e'_{i_k} \otimes a_\alpha.$$

For n-dimensional ordinary Lie algebra acting in p-dimensional module

$$\dim C^k = p \binom{n}{k},$$

for $(n|m)$-dimensional Lie superalgebra

$$\dim C^k = p \sum_{i=0}^{k} \binom{n}{k-i} \binom{m+i-1}{i} \equiv p \binom{n}{k} + p \sum_{i=1}^{k} \binom{n}{k-i} \binom{m+i-1}{i}.$$

Scheme of splitting algorithm

$$C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1}$$

$$\Downarrow$$

$$C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} = \bigoplus_{g\in G} C_g^{k-1} \xrightarrow{d_g^{k-1}} C_g^k \xrightarrow{d_g^k} C_g^{k+1}$$

$G \subseteq \mathbb{Z}$ – integer grading

$$\Downarrow$$

$$C_g^{k-1} \xrightarrow{d_g^{k-1}} C_g^k \xrightarrow{d_g^k} C_g^{k+1} = \bigoplus_{s\in S} C_{g,s}^{k-1} \xrightarrow{d_{g,s}^{k-1}} C_{g,s}^k \xrightarrow{d_{g,s}^k} C_{g,s}^{k+1}$$

S – finite or infinite set of subcomplexes

$$C_g^i = \bigoplus_{s\in S} C_{g,s}^i$$

$$d_g^i = \bigoplus_{s\in S} d_{g,s}^i \Longleftrightarrow d_g^i \text{ – block matrices}$$

$$\Downarrow$$

$$H_{g,s}^k = \text{Ker } d_{g,s}^k / \text{Im } d_{g,s}^{k-1}.$$

Lie superalgebras with antibrackets (odd Poisson structure)

Odd symplectic structure:
$x^1, \ldots, x^n; \theta_1, \ldots, \theta_n$ – even and odd (grassmann) variables;
$\sum_{i=1}^n dx^i \wedge d\theta_i$ – invariant 2-form;
$f(x^1, \ldots, \theta_n), g(x^1, \ldots, \theta_n)$ – generating functions (hamiltonians).

$\{f, g\}_{Bb} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta_i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial x^i} \right)$ – *Buttin brackets* (antibrackets, odd Poisson brackets).

Buttin algebra B(n).

$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x^i \partial \theta_i}$ – operator of *master equation* in Batalin–Vilkovisky method.

Special Buttin algebra SB(n) satisfies to the divergence free condition $\Delta f = 0$.

Special Leites algebra is the qoutient of SB(n) w.r.t. center Z, i. e.

$$\mathrm{SLe(n)} = \mathrm{SB(n)}/Z.$$

The case of even dimension $n = 2m$ is more interesting for physics.

Lie superalgebra SLe(2)

Variables: even x, y, $\mathrm{gr}(x) = \mathrm{gr}(y) = 1$; odd θ, ψ, $\mathrm{gr}(\theta) = \mathrm{gr}(\psi) = -1$.

Basis elements and nonzero commutators:

$\|-2\|$	O_1	$=$	$\theta\psi$
$\|-1\|$	E_2	$=$	θ
$\|-1\|$	E_3	$=$	ψ
$\|0\|$	E_4	$=$	$y\theta$
$\|0\|$	E_5	$=$	$y\psi - x\theta$
$\|0\|$	E_6	$=$	$x\psi$
$\|1\|$	O_7	$=$	y
$\|1\|$	O_8	$=$	x
$\|1\|$	E_9	$=$	$y^2\theta$
$\|1\|$	E_{10}	$=$	$y^2\psi - 2xy\theta$
$\|1\|$	E_{11}	$=$	$xy\psi - \frac{1}{2}x^2\theta$
$\|1\|$	E_{12}	$=$	$x^2\psi$
$\|2\|$			$\vdots$

(1)	$[E_2, E_5]$	$=$	E_2
(2)	$[E_2, E_6]$	$=$	$-E_3$
(3)	$[E_3, E_4]$	$=$	$-E_2$
(4)	$[E_3, E_5]$	$=$	$-E_3$
(5)	$[E_5, E_4]$	$=$	$-2E_4$
(6)	$[E_6, E_4]$	$=$	E_5
(7)	$[E_5, E_6]$	$=$	$2E_6$
(8)	$[O_1, O_7]$	$=$	$-E_2$
(9)	$[E_5, O_7]$	$=$	$-O_7$
(10)	$[E_6, O_7]$	$=$	$-O_8$
(11)	$[O_1, O_8]$	$=$	E_3
(12)	$[E_4, O_8]$	$=$	$-O_7$
(13)	$[E_5, O_8]$	$=$	O_8
		$\vdots$	

Important subalgebras:

$A_{\leq 0} = \mathrm{span}\{O_1, E_2, \ldots, E_6\}$ – non-positive subalgebra,

$A_{<0} = \mathrm{span}\{O_1, E_2, E_3\}$ – negative subalgebra,

$A_0 = \mathrm{span}\{E_4, E_5, E_6\} \sim \mathrm{so}(3) \sim \mathrm{sl}(2) \sim \mathrm{sp}(2)$ – zero subalgebra,

$A_{\leq 0} = A_{<0} \ni\!\!\!\!+ A_0$ – semidirect sum of commutative ideal and simple algebra.

Computing $H^k_g(\mathrm{SLe}(2))$ $(k,g) \in [1,\ldots,10] \otimes [-2k,\ldots,-2k+16]$ by the C program ***LieCohomology***

$k \backslash g{+}2k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1 1 $\boxed{a}$ 1	2 2 1	3 3 1	6 6 1	8 8 1	10 10 1	12 12 1	14 14 1	16 16 1	18 18 1	20 20 1	22 22 1	24 24 1	26 26 1	28 28 1	30 30 1	32 32 1
2	1 1 $\boxed{a^2}$ 1	2 2 1	4 4 1	12 8 3	23 13 $\boxed{b}$ 4	44 16 6	73 22 8	116 26 10	171 30 14	244 34 18	333 38 23	444 42 28	575 46 34	732 50 40	913 54 47	1124 58 54	1363 62 62
3	1 1 $\boxed{a^3}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	118 26 15	226 34 23	414 39 36	718 48 52	1182 54 80	1870 60 119	2858 66 176	4224 72 241	6082 78 330	8552 84 434	11766 90 570
4	1 1 $\boxed{a^4}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	121 26 15	246 34 23	491 42 41	952 52 71	1780 65 124	3204 72 197	5584 84 311	9398 92 489	15343 100 787	24348 108 1187	37649 116 1776
5	1 1 $\boxed{a^5}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	121 26 15	246 34 23	492 42 41	970 52 71	1867 65 $\boxed{c}$ 124	3528 76 197	6546 88 358	11878 104 606	21073 113 1009	36540 130 1578	61884 140 2556
6	1 1 $\boxed{a^6}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	121 26 15	246 34 23	492 42 41	970 52 71	1867 65 124	3534 76 197	6605 88 358	12162 104 606	22102 118 1009	39652 134 1598	70110 153 $\boxed{d}$ 2802
7	1 1 $\boxed{a^7}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	121 26 15	246 34 23	492 42 41	970 52 71	1867 65 124	3534 76 197	6605 88 358	12162 104 606	22119 118 1009	39796 134 1598	70817 153 $\boxed{ad}$ 2802
8	1 1 $\boxed{a^8}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	121 26 15	246 34 23	492 42 41	970 52 71	1867 65 124	3534 76 197	6605 88 358	12162 104 606	22119 118 1009	39796 134 1598	70817 153 2802
9	1 1 $\boxed{a^9}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	121 26 15	246 34 23	492 42 41	970 52 71	1867 65 124	3534 76 197	6605 88 358	12162 104 606	22119 118 1009	39796 134 1598	70817 153 2802
10	1 1 $\boxed{a^{10}}$ 1	2 2 1	4 4 1	12 8 3	26 13 4	56 18 7	121 26 15	246 34 23	492 42 41	970 52 71	1867 65 124	3534 76 197	6605 88 358	12162 104 606	22119 118 1009	39796 134 1598	70817 153 2802

$a = c(\theta\psi)$, $b = c(x,\theta) = c(y,\psi)$; $ab = 0$, $ac = 0$, $a^2 d = 0$.

$\dim C^5_{4,s} = 387$ $\dim C^6_{4,s} = 912$ $\dim C^7_{4,s} = 1847$ $\dim Z^6_{4,s} = 286$ $\dim B^6_{4,s} = 285$ $\dim H^6_{4,s} = 1$

For (k,g) = (6,4) = (6, -2*6+16) time = 1 h 32 min 20.07 sec on PC Pentium 3, 667MHz

Competitive projects

1. *Homology Package*, N. van den Hijligenberg, G. Post, in *Reduce,*

2. *SuperLie*, P. Grozman, D. Leites, in *Mathematica.*

Comparison with *SuperLie* for $H_g^k(\mathrm{SH}(0|4))$

$k\backslash g$	-8	-6	-4	-2	0	2	4	6	8
1									
2				$*$	$*$	$*$			
3					$*$				
4			$*$	$*$	$*$	$*$	$*$		
5				$*$	$*$	$*$			
6		$*$	$*$	$*$	$*$	$*$	$*$	$*$	
7			$*$	$*$	$*$	$*$	$*$		
8	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Rows 1–4: *SuperLie*

Rows 5–8: *LieCohomology*, all tasks within several seconds on Pentium 3, 667 MHz