A review of fast circle and helix fitting

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Outline

- Introduction and background
- Circle fitting
- Helix fitting
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Introduction and background

- In a **detector embedded in a homogeneous magnetic field**, the particle trajectories are **helices**.
- Examples: Inner trackers in CMS and ATLAS.
- LHC track reconstruction methods have to be **precise** and **fast**
- The **method of choice** will very likely depend on the **requirements of the actual physics analysis**.

Introduction and background

Track fitting methods can roughly be divided into **two separate categories**:

- **Precise and slow**
- **Approximate and fast**

Those of the **latter category mainly work for 2D data** — i.e. data either coming from a **2D detector** or **projected data from a 3D detector**

Introduction and background

The global least-squares method:

- Used for **many decades** in **HEP experiments**.
- **Proper treatment** of **elastic, multiple Coulomb scattering** included in the method **during the 70's**.
- **Close to optimal** in precision, but may be **computationally quite expensive** with a **large number of measurements** and/or a **large number of scattering devices**.

Introduction and background

The Kalman filter:

- **Recursive least-squares estimation.**
- Therefore **suitable for combined track finding and fitting**
- **Equivalent** to **global least-squares method** including all **correlations between measurements** due to **multiple scattering**.
- Probably the **most widely used method today**.

Introduction and background

Both the global LS fit and the Kalman filter may need **previous knowledge** of the track:

- as an expansion point (reference track) of the linearization procedure,
- for the computation of the multiple scattering covariance matrix.

This is **particularly important** for tracks with **large curvature (low momentum)**. Therefore **fast preliminary fits** are required.

Circle fitting

Some specialized methods for circle fitting:

- **Conformal mapping** — maps **circles through the origin onto straight lines**
- **Karimäki method** — based on an **approximate, explicit solution** to the **non-linear problem of circle fitting**
- **Riemann fit** — maps **circles onto planes in space**, results in **exact linear fit in 3D**

Circle fitting

Conformal mapping [1]

Inversion in the complex plane:

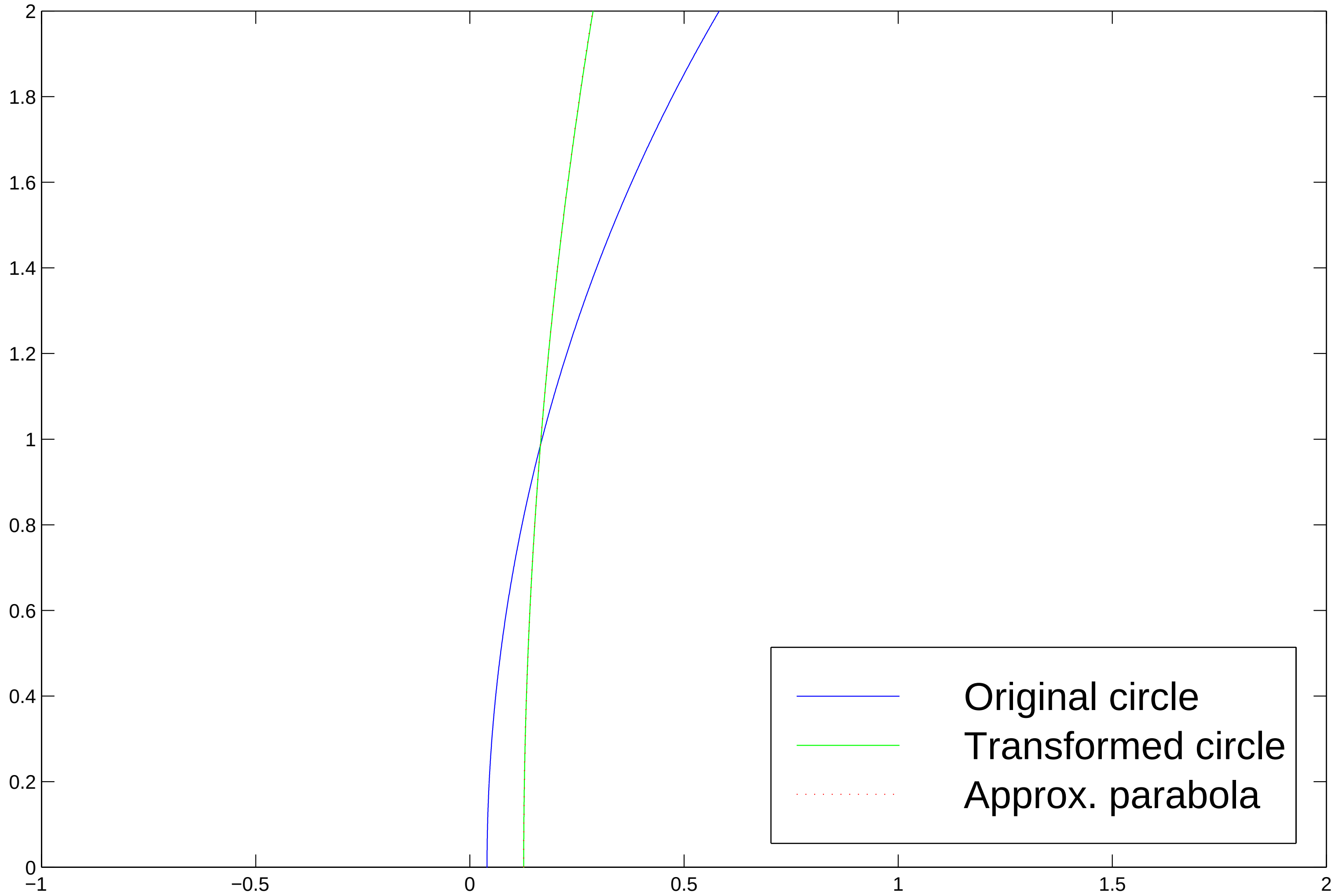
$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{y}{x^2 + y^2}$$

- A **circle through the origin** is mapped on a **straight line**.
- the **impact parameter** of the line is inversely proportional to the radius of the circle.

Circle fitting

- A circle with **small impact parameter** is mapped on a circle with **small curvature** (proportional to the impact parameter to first order).
- The latter circle can be approximated by a **parabola**.
- Fast, linear fit of the coefficients of the parabola.

Circle fitting



Circle fitting

Karimäki method [2]

Under the assumption that the **impact parameter is small compared to the radius**:

$$|\epsilon| \ll \rho$$

an **explicit solution** to the non-linear problem can be found. An additional correction procedure gives very good final precision.

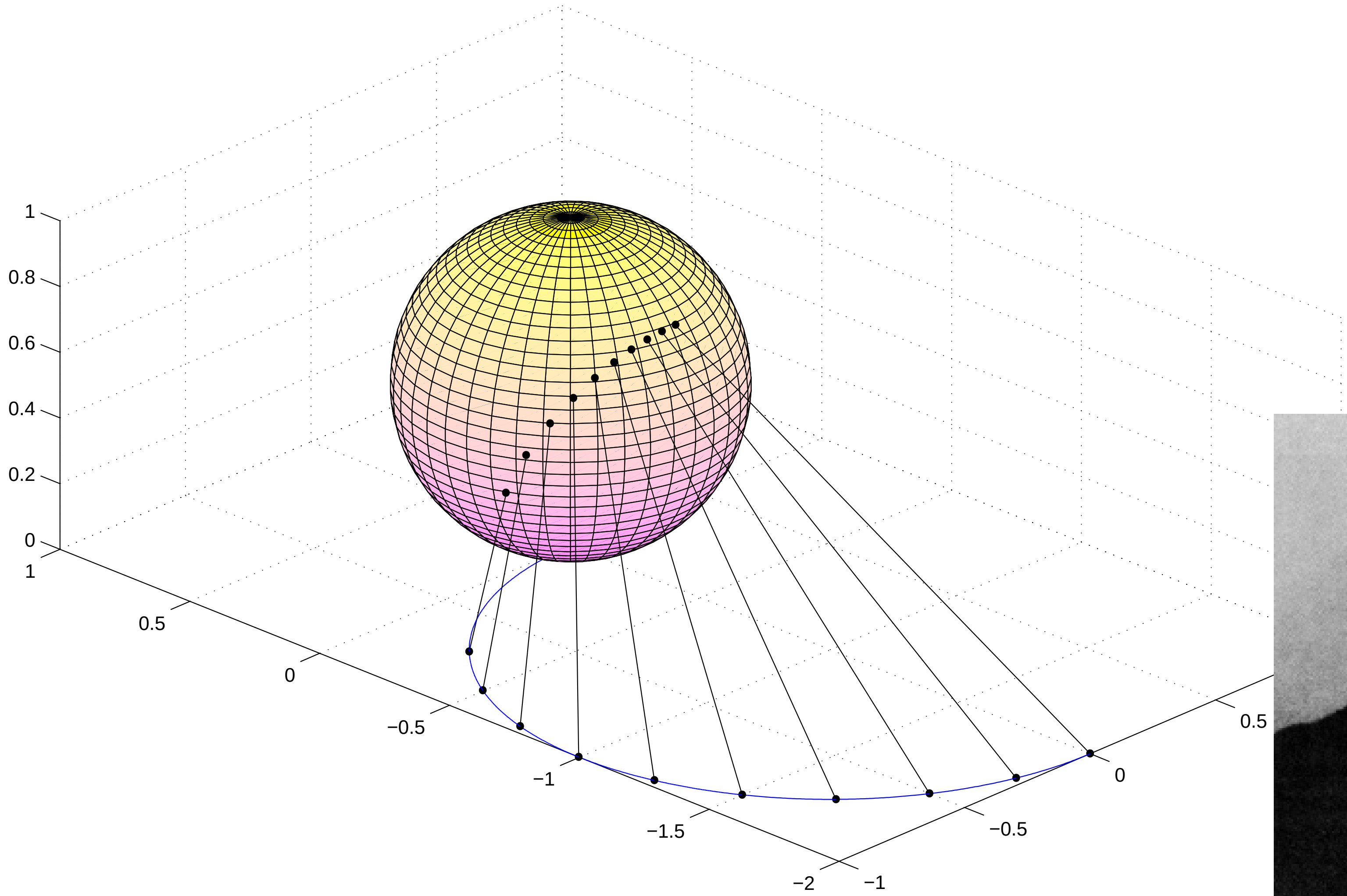
Circle fitting

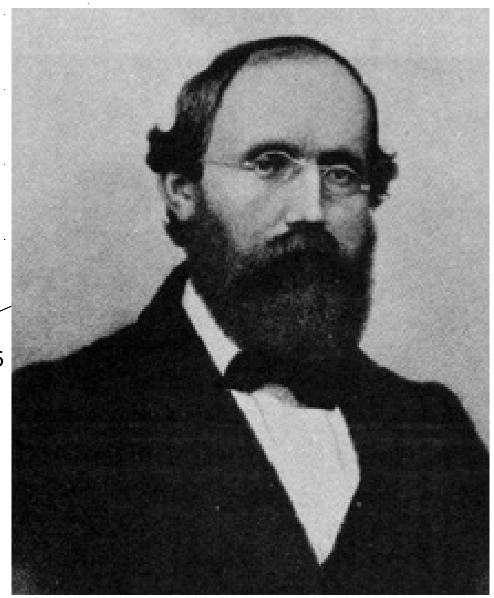
The Riemann fit I [3]

Based on a **conformal mapping (stereographic projection)** of 2D-measurements to 3D-points on the **Riemann sphere**:

$$
\begin{aligned}
x_i &= R_i \cos \Phi_i / (1 + R_i^2) \\
y_i &= R_i \sin \Phi_i / (1 + R_i^2) \\
z_i &= R_i^2 / (1 + R_i^2)
\end{aligned}
$$

Circle fitting





Circle fitting

- **Circles and lines** in the plane map uniquely onto **circles** on the Riemann sphere.
- Since a circle on the Riemann sphere uniquely defines a plane in space, there is a **one-to-one correspondence** between **circles and lines** in the plane and **planes in space**.

The Riemann fit II [5]

Non-conformal mapping of 2D-measurements to 3D-points on a **cylindrical paraboloid**:

$$
\begin{aligned}
x_i &= R_i \cos \Phi_i \\
y_i &= R_i \sin \Phi_i \\
z_i &= R_i^2
\end{aligned}
$$

This mapping is even simpler.

Circle fitting

- Again, **points on a circle** are mapped on **points lying on a plane** (but not on a circle).
- Thus, the task of **fitting circular arcs** in the plane is transformed into the task of **fitting planes** in space.
- This can be done in a **fast** and **non-iterative** manner.
- Moreover, there is **no need** for any **track parameter initialization**.

Circle fitting

- A plane can be defined by a **unit length normal vector** $\boldsymbol{n}^T = (n_1, n_2, n_3)$ and a **signed distance** c from the origin.
- **Fitting a plane** to N measurements on the sphere or paraboloid requires **finding the minimum** of

$$S = \sum_{i=1}^{N} (c + n_1 x_i + n_2 y_i + n_3 z_i)^2 = \sum_{i=1}^{N} d_i^2$$

with respect to $\{c, n_1, n_2, n_3\}$.

Circle fitting

- The minimum of S is found by choosing $\boldsymbol{n}$ to be the **eigenvector** to the **smallest eigenvalue** of the **sample covariance matrix** of the measurements.
- The distance c is given by the fact that the **fitted plane passes through the mean vector of the measurements**.
- The fitted parameters can then be transformed back to the circle parameters in the plane.

Circle fitting

- The precision and the speed of the Riemann fit (RF) has been assessed by a comparison with
  - a **non-linear least-squares fit** (NLS),
  - a **global linearized least-squares fit** (GLS),
  - the **Kalman filter** (KF),
  - and the **conformal mapping** (CM).
- We show results from a **simulation experiment** in the ATLAS Transition radiation Tracker, with about 35 observations per track.

Circle fitting

Method	V_{rel}	t_{rel}
NLS w/o initialization	1.000	36.3
NLS with initialization	1.000	41.4
GLS w/o initialization	1.001	15.9
GLS with initialization	1.001	21.1
KF w/o initialization	1.001	28.2
KF with initialization	1.001	33.3
CM (parabola fit)	1.582	1.03
RF (circle fit)	1.003	1.00

Red=Baseline

Circle fitting

- The RF can be **corrected for the non-orthogonal intersection** of the track with the detectors. This is important for **low-momentum tracks**, but requires an iteration.
- Formulas for the **covariance matrix** of the fitted parameters have been derived [4].

Circle fitting

The RF can also deal with **multiple scattering** [5]:

- The **cost function** is **generalized**:

$$S = \boldsymbol{d}^T \boldsymbol{V}^{-1} \boldsymbol{d}$$

- $\boldsymbol{d}$ is the **vector containing the distances** from the **measurements to the plane**.

- $\boldsymbol{V}$ is an **approximate covariance matrix** of these distances **including correlations from multiple scattering**.

Circle fitting

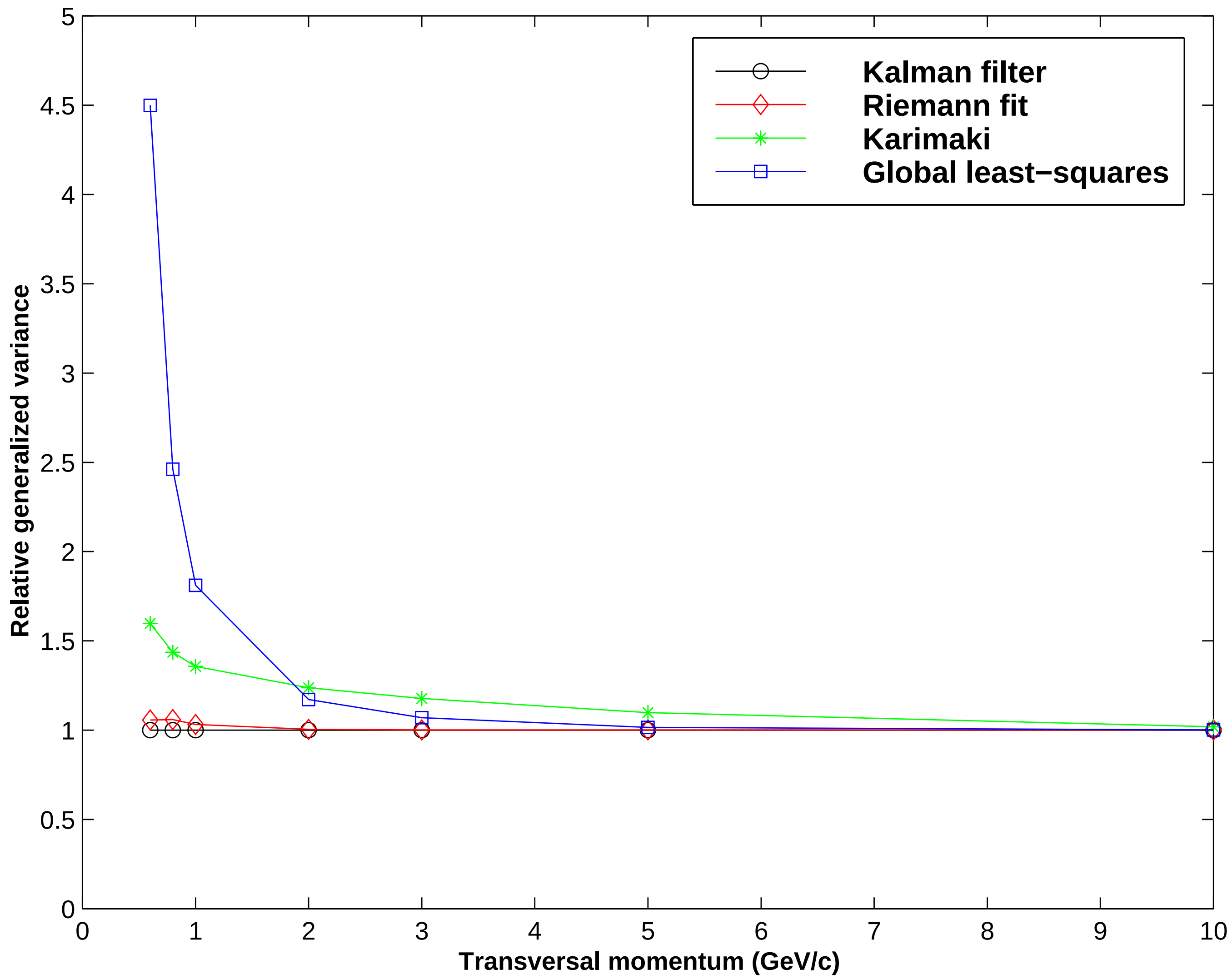
- Again, the **minimum** of S with respect to the **plane parameters** defines the **fitted plane**.
- The **normal vector of the plane** is found in a **similar manner as before** — only the **building-up** of the **sample covariance matrix of the measurements** is **slightly modified**.
- It is **not straightforward** to **generalize any of the other circle estimators** (**conformal mapping, Karimäki**) in this way.

Circle fitting

We have performed a **simulation experiment** in the **ATLAS Inner Detector TRT**. **Four methods** have been compared:

- The **generalized Riemann fit**
- The **Kalman filter**
- The **Karimäki method** including the diagonal terms of the covariance matrix
- The **global least-squares fit** without contributions from multiple scattering in the covariance matrix

Circle fitting



Helix fitting

The circle fit can be **extended to a helix fit** by using the **linear relation** between the arc length s and z [6].

- After the circle fit, the arc length between successive observations is computed.
- A regression of z on s (barrel) or of s on z (forward) gives the polar angle θ plus an additional coordinate.
- In disk type detectors the radial positions of the hits are predicted from the line fit, and the entire procedure is repeated.

Helix fitting

We have done a **simulation experiment** in a simplified model of the **CMS Tracker**. **Three methods** have been compared:

- **Riemann Helix fit** based on Riemann circle fit (RHF)
- **Kalman filter** (KF)
- **Global least-squares fit** (GLS)

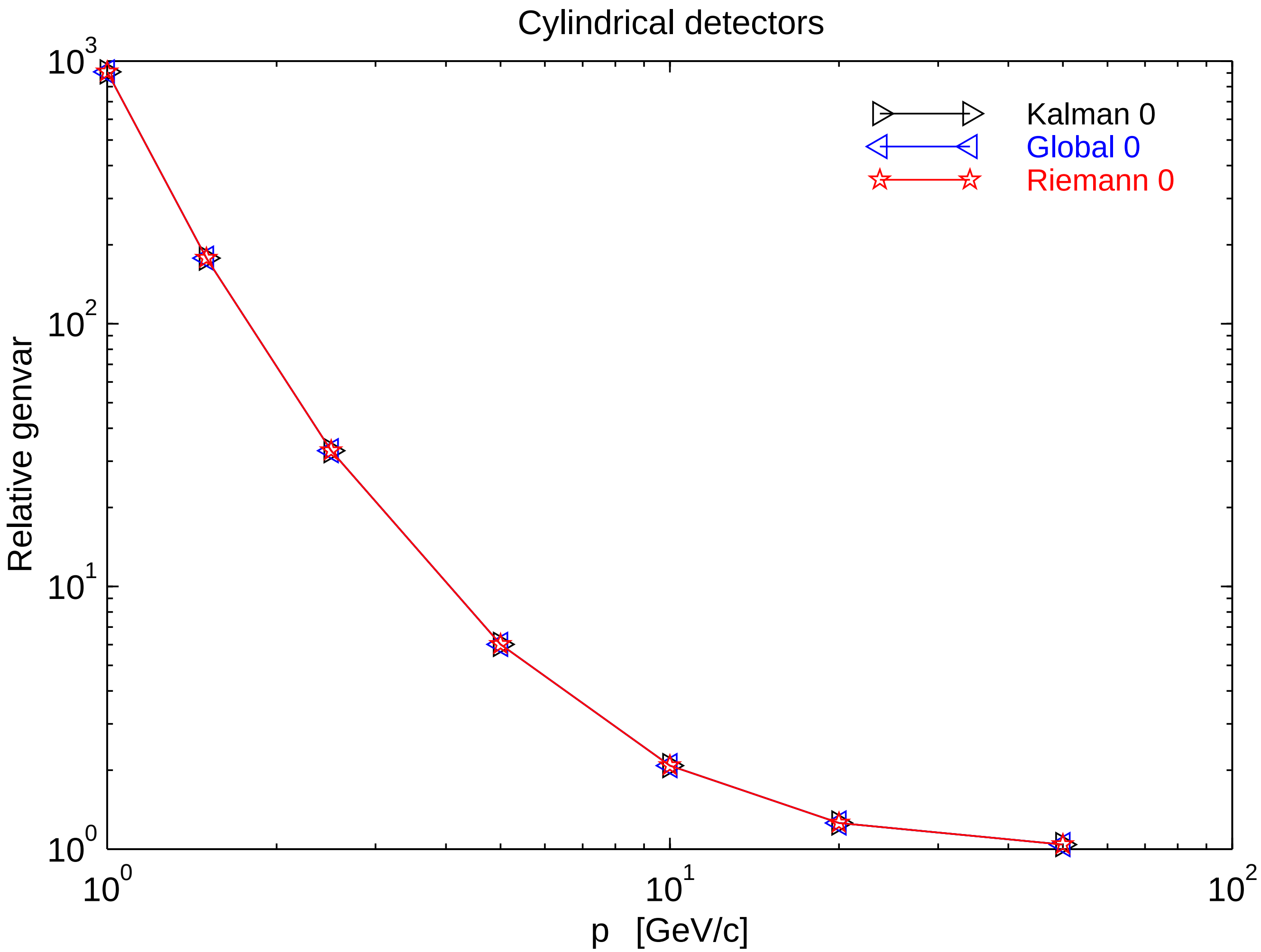
Helix fitting

Multiple Scattering has been treated on different levels:

Level	Covariance matrix of multiple scattering	Applies to
0	None	All methods
1	Approximate	GLS, RHF
2	Exact, but no correlations between projections	GLS, RHF
3	Exact, including all correlations	GLS, KF

When required (KF or level>0), a reference track has been computed by a preliminary RHF.

Helix fitting



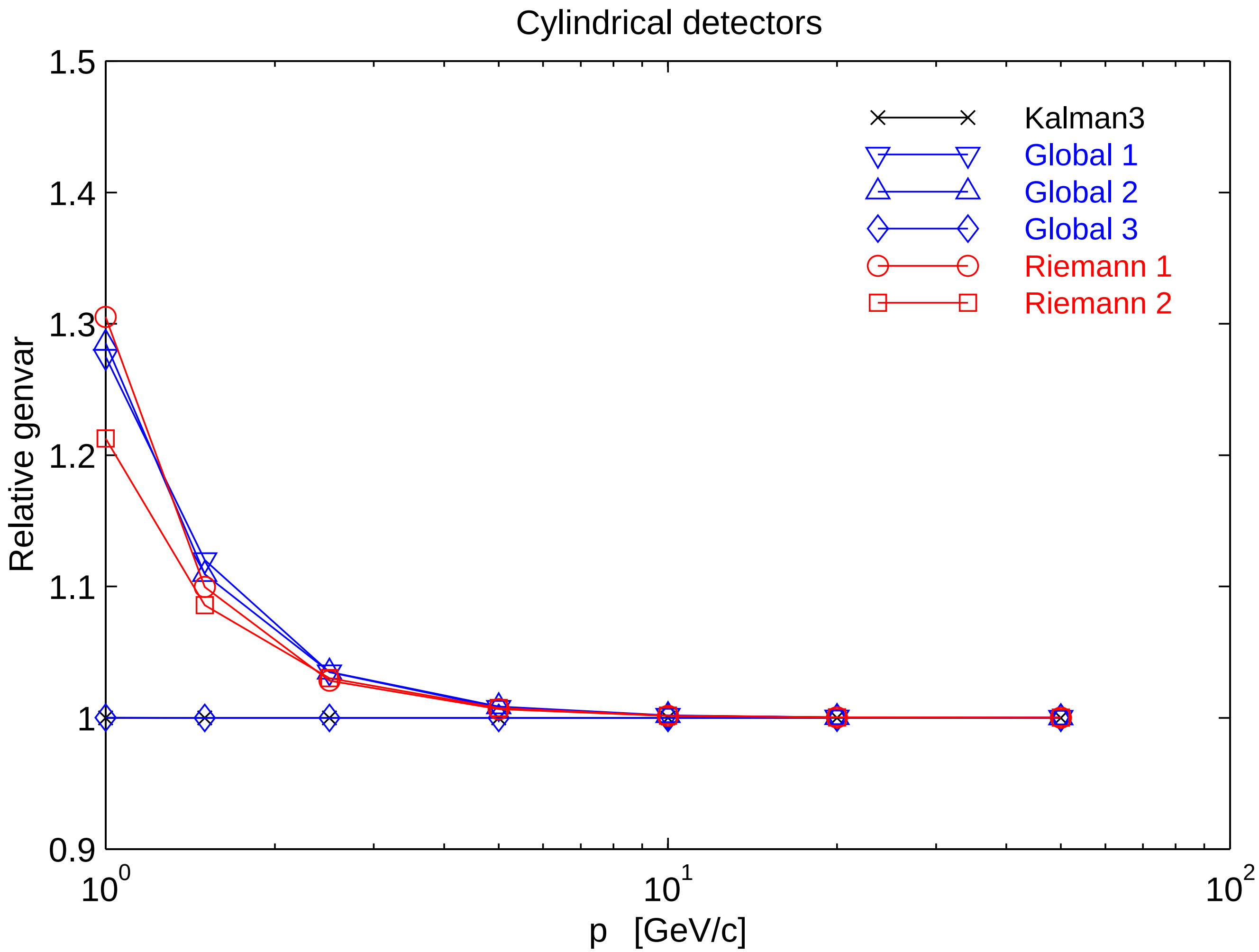
Generalized variance on level 0, relative to the KF on level 3

Helix fitting

Method	Level	t_{rel}
Kalman filter	0	0.98
Global fit	0	0.97
Riemann fit	0	0.70

Timing on level 0, relative to the KF on level 3

Helix fitting



Generalized variance on level>0, relative to the KF on level 3

Helix fitting

Method	Level	t_{rel}
Kalman filter	3	1.00
Global fit	1	1.08
Global fit	2	1.38
Global fit	3	1.38
Riemann fit	1	0.84
Riemann fit	2	1.16

Timing on level>0, relative to the KF on level 3

Helix fitting

- These results have been obtained from the **C++ implementation**. The program is available from the authors on request.
- For **disk detectors** the Riemann helix fit is **not competitive as an exact fit**, because of the need to **iterate**, but still **highly suitable** as a **preliminary fit** for the KF or the GLS.

Conclusions

- In the absence of multiple scattering, the Riemann circle fit is virtually as precise as either non-linear or linear least-squares estimators and much faster
- In the presence of multiple scattering, the Riemann circle fit is as precise as the Kalman filter over a large range of momentum and superior in precision to similar methods (Karmäki, Conformal Mapping)

Conclusions

- The Riemann helix fit is a viable alternative to conventional least-squares fits, especially if multiple scattering can be neglected.
- It is highly suitable as a fast approximate fit for generating a reference track for the Kalman filter or the global least-squares fit.

References

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