Analytical Evaluation of Certain On-Shell Two-Loop Three-Point Diagrams

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Based on common work with
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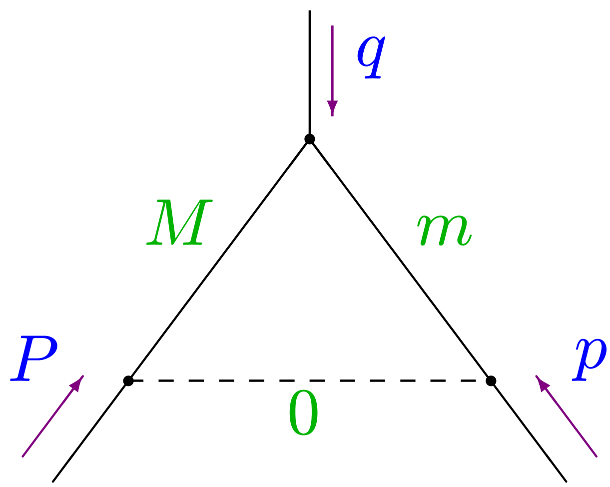
One-loop three-point function



All external momenta are ingoing, $P+p+q=0$.

On-shell conditions: $P^2=M^2, \quad p^2=m^2$.

$$J \equiv \int \frac{\mathrm{d}^n r}{r^2 \left[(P+r)^2-M^2\right] \left[(p-r)^2-m^2\right]} \Bigg|_{\substack{P^2=M^2 \\ p^2=m^2}} ,$$

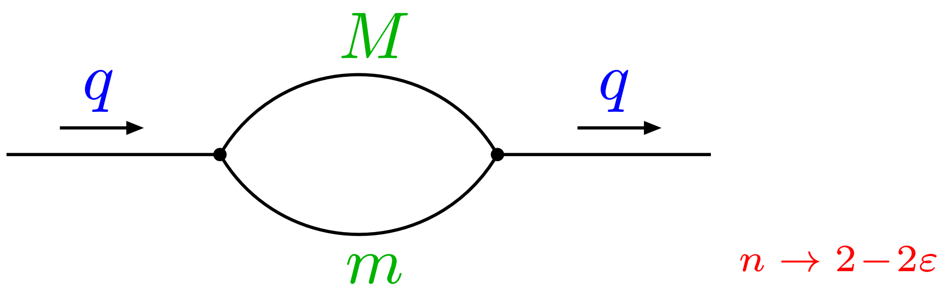
where $n=4-2\varepsilon$ is the space-time dimension.

Reduction to a 2-point function

A.I. Davydychev and M.Yu. Kalmykov, Nucl. Phys. **B605** (2001) 266

In this on-shell limit, the 3-point function reduces to a 2-point function with the external momentum q, masses m and M, unit powers of the propagators, and the space-time dimension $2-2\varepsilon$,

$$J = \frac{\pi}{2\varepsilon}\cdot$$



To get result for J, we need to expand the 2-point function up to the next term of the ε expansion.

Log-sine function and Nielsen polylogarithm

Any term of the ε expansion can be calculated in terms of the log-sine functions

$$\mathsf{Ls}_j(\theta) = -\int_0^\theta \mathrm{d}\phi \, \ln^{j-1}\left|2\sin\frac{\phi}{2}\right| ,$$

whose analytic continuation yields Nielsen polylogarithms

$$S_{a,b}(z) = \frac{(-1)^{a+b-1}}{(a-1)!\, b!} \int\limits_0^1 \mathsf{d}\xi \, \frac{\ln^{a-1}\xi \ln^b(1-z\xi)}{\xi} .$$

In particular,

$$S_{a,1}(z) = \mathsf{Li}_{a+1}(z) , \quad S_{0,b}(z) = \frac{(-1)^b}{b!} \ln^b(1-z).$$

Higher terms of the ε expansion, as well as their analytic continuation, are given in

A.I. Davydychev, Phys. Rev. **D61** (2000) 087701;

A.I. Davydychev and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283.

One-loop result

Expanding exact result in ε and m^2, we obtain

$$
\begin{aligned}
J\big|_{m\to 0} \;=\;& \mathrm{i}\pi^{2-\varepsilon}\Gamma(1+\varepsilon)(M^2)^{-1-\varepsilon}\sigma \\
& \times\left\{ -\frac{1}{2\varepsilon}\left[\ln\frac{m^2}{M^2} + 2\ln\sigma\right] \right. \\
& \left. +\frac{1}{4}\ln^2\frac{m^2}{M^2} - \frac{1}{2}\ln^2\sigma + \mathrm{Li}_2\left(1-\sigma\right) \right\} \\
& +\mathcal{O}(\varepsilon, m^2\ln^2(m^2)),
\end{aligned}
$$

where

$$
\sigma \equiv \frac{M^2}{M^2-q^2}.
$$

One-loop result: the $m \to 0$ limit

If we are interested in the case $m = 0$, we cannot use this result, since there are $\ln m$ singularities. To regulate them, we need to return to the exact dimensionally-regulated representation and put $m = 0$:

$$J|_{m=0} = -\frac{\mathrm{i}\pi^{2-\varepsilon}\Gamma(1+\varepsilon)}{(M^2-q^2)^{1+\varepsilon}} \frac{1}{2\varepsilon^2} \, {}_2F_1\left(\begin{matrix} -\varepsilon, 1+\varepsilon \\ 1-\varepsilon \end{matrix} \middle| 1-\sigma \right),$$

where ${}_2F_1$ is the Gauss hypergeometric function. Expanding in ε we get

$$J|_{m=0} = \mathrm{i}\pi^{2-\varepsilon}\Gamma(1+\varepsilon)(M^2)^{-1-\varepsilon}\sigma \times \left\{ -\frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon}\ln\sigma - \frac{1}{2}\ln^2\sigma - \mathrm{Li}_2\,(1-\sigma) \right\} + \mathcal{O}(\varepsilon).$$

One-loop result: other contributions

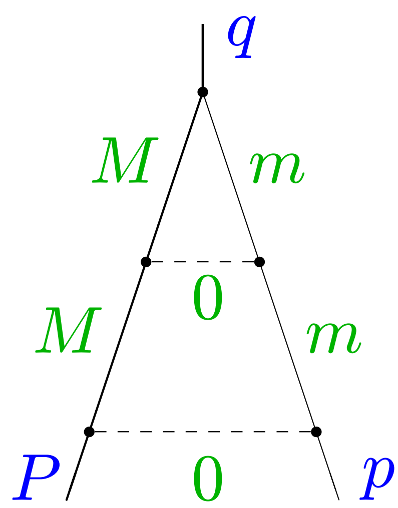
M. Beneke and V.A. Smirnov, Nucl. Phys. **B522** (1998) 321;
V.A. Smirnov, Phys. Lett. **B465** (1999) 226

Adding (2c) contribution,

$$\frac{\mathrm{i}\pi^{2-\varepsilon}\Gamma(1+\varepsilon)}{M^{2\varepsilon}(M^2-q^2)}\,\frac{1}{2\varepsilon^2}\left(\frac{m^2}{M^2}\right)^{-\varepsilon}$$
$$=\frac{\mathrm{i}\pi^{2-\varepsilon}\Gamma(1+\varepsilon)}{M^{2\varepsilon}(M^2-q^2)}\left\{\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}\ln\frac{m^2}{M^2}+\frac{1}{4}\ln^2\frac{m^2}{M^2}\right\}+\mathcal{O}(\varepsilon),$$

we obtain nothing but the expanded exact answer.

Two-loop diagram with two masses



On-shell conditions: $P^2 = M^2, \quad p^2 = m^2$.

Applications:

- QED correction to muon decay
- QCD correction to $t \to W^+ b$
- QCD correction to $t \to H^+ b$

Expansion by regions

$$F(q^2, M^2, m^2; \varepsilon)$$
$$= \iint \frac{\mathrm{d}^n k\ \mathrm{d}^n l}{(l^2-2Pl)(l^2-2pl)(k^2-2Pk)(k^2-2pk)k^2(k-l)^2}\,.$$

We are interested in the case when $m^2 \ll \{M^2, |q^2|\}$.

Choose $n_{1,2} = (\frac{1}{2}, 0, 0, \mp\frac{1}{2})$, ($2n_{1,2}k = k_\pm \equiv k_0 \pm k_3$), $\underline{k} = (k_1, k_2)$, $P = (M, \vec{0})$ and $p = \alpha n_1 + \frac{m^2}{\alpha} n_2$.

The relevant regions are:

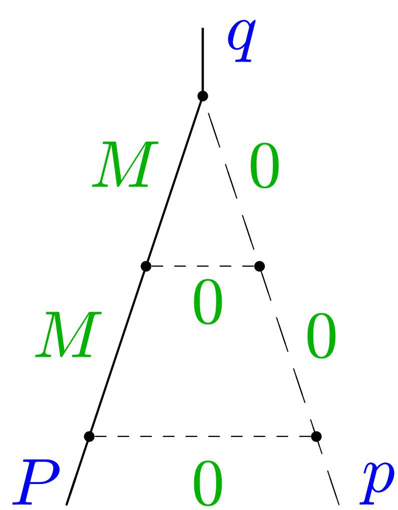
hard (h),	$k \sim M\,;$
1-collinear (1c),	$k_+ \sim m^2/M,\ k_- \sim M\,,\ \underline{k} \sim m\,;$
2-collinear (2c),	$k_+ \sim M,\ k_- \sim m^2/M\,,\ \underline{k} \sim m\,;$
ultrasoft (us),	$k \sim m^2/M\,.$

List of regions (k, l) generating non-zero contributions to the expansion of F in the leading order:
(h-h), (1c-h), (1c-1c) and (us-1c).

V.A. Smirnov, Phys. Lett. **B465** (1999) 226

Two-loop planar diagram with $m = 0$

The (h-h) region generates Taylor expansion of the integrand in m^2. In the leading order, this is just the diagram with $m = 0$:



On-shell conditions: $P^2 = M^2, \quad p^2 = 0$.

Integration by parts?

F.V. Tkachov, Phys. Lett. **B100** (1981) 65

K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B192** (1981) 159

Other approaches?

J. Fleischer, A.V. Kotikov and O.L. Veretin, Nucl. Phys. **B547** (1999) 343

Mellin–Barnes representation ($m = 0$)

Contour integral representation for the vertex with $m = 0$ and unit powers of all propagators:

$$-\frac{\pi^{4-2\varepsilon}(M^2)^{-2-2\varepsilon}}{\Gamma(1-2\varepsilon)} \frac{1}{(2\pi \mathrm{i})^4} \int\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\int\int \mathrm{d}z \, \mathrm{d}\tilde{z} \, \mathrm{d}t \, \mathrm{d}u \, \sigma^{2+2\varepsilon+z+\tilde{z}}$$

$$\times \frac{\Gamma(-t)\,\Gamma(-u)}{\Gamma(1-t)\,\Gamma(1-u)} \Gamma(-z)\,\Gamma(-\tilde{z})$$

$$\times \Gamma(1+\varepsilon+t+u+z)\,\Gamma(1+\varepsilon-t-u+\tilde{z}) \frac{\Gamma(1+t+u)}{\Gamma(1-2\varepsilon+t+u)}$$

$$\times \Gamma(-\varepsilon-u+z)\,\Gamma(-\varepsilon+u+\tilde{z})\,\Gamma(-\varepsilon-t-z)\,\Gamma(-\varepsilon+t-\tilde{z}) .$$

Contour integrals are chosen so as to separate the right and left series of poles of the Γ functions in all four variables z, $\tilde{z}$, t and u. For small negative values of ε, this condition can be satisfied even by straight contours (parallel to the imaginary axes), if we choose, say, $\mathrm{Re}z = \mathrm{Re}\tilde{z} = \frac{1}{2}\varepsilon$, $\mathrm{Re}t = \varepsilon$ and $\mathrm{Re}u = \frac{1}{4}\varepsilon$.

How to extract singularities, etc.?

Result of the calculation ($m = 0$)

$$\pi^{4-2\varepsilon} \mathrm{e}^{-2\gamma_{\mathrm{E}}\varepsilon} (M^2)^{-2-2\varepsilon} \sigma^{2+4\varepsilon}$$

$$\times \left\{ \frac{1}{12\varepsilon^4} + \frac{1}{\varepsilon^2} \left[\frac{\pi^2}{12} - \frac{1}{4} \ln^2 \sigma - \frac{1}{2} \mathsf{Li}_2 (1-\sigma) \right] \right.$$

$$+ \frac{1}{\varepsilon} \left[\frac{91\zeta_3}{36} + \frac{5}{12} \ln^3 \sigma + \frac{3}{2} \ln \sigma \; \mathsf{Li}_2 (1-\sigma) \right.$$

$$\left. + \frac{1}{2} S_{1,2}(1-\sigma) - \frac{3}{2} \mathsf{Li}_3 (1-\sigma) \right]$$

$$+ \frac{179\pi^4}{1440} - \frac{7\pi^2}{24} \ln^2 \sigma - \frac{19}{48} \ln^4 \sigma - \frac{9}{4} \ln^2 \sigma \; \mathsf{Li}_2 (1-\sigma)$$

$$- \frac{7\pi^2}{12} \mathsf{Li}_2 (1-\sigma) - \frac{7}{2} \ln \sigma \; S_{1,2}(1-\sigma) + \frac{5}{2} \ln \sigma \; \mathsf{Li}_3 (1-\sigma)$$

$$- \left[\mathsf{Li}_2 (1-\sigma) \right]^2 - \frac{13}{2} S_{1,3}(1-\sigma) + \frac{7}{2} S_{2,2}(1-\sigma)$$

$$\left. - \frac{5}{2} \mathsf{Li}_4 (1-\sigma) + \mathcal{O}(\varepsilon) \right\},$$

$$\text{where} \qquad \sigma \equiv \frac{M^2}{M^2 - q^2}.$$

Collecting all contributions

$$\pi^{4-2\varepsilon}\mathrm{e}^{-2\gamma_{\mathrm{E}}\varepsilon}(M^2)^{-2-2\varepsilon}\sigma^2\left\{\frac{1}{8\varepsilon^2}\left(\ln\frac{m^2}{M^2}+2\ln\sigma\right)^2\right.$$

$$-\frac{1}{\varepsilon}\left[\frac{1}{6}\ln^3\frac{m^2}{M^2}+\frac{1}{2}\ln^2\frac{m^2}{M^2}\ln\sigma+\left(\frac{\pi^2}{12}+\frac{1}{4}\ln^2\sigma\right)\ln\frac{m^2}{M^2}\right.$$

$$\left.+\frac{1}{2}\left(\ln\frac{m^2}{M^2}+2\ln\sigma\right)\mathrm{Li}_2\left(1-\sigma\right)+\frac{\pi^2}{6}\ln\sigma-\frac{1}{6}\ln^3\sigma+\zeta_3\right]$$

$$+\frac{13}{96}\ln^4\frac{m^2}{M^2}+\frac{5}{12}\ln^3\frac{m^2}{M^2}\ln\sigma+\frac{5\pi^2}{16}\ln^2\frac{m^2}{M^2}+\frac{3}{8}\ln^2\frac{m^2}{M^2}\ln^2\sigma$$

$$+\frac{3}{2}\zeta_3\ln\frac{m^2}{M^2}+\frac{1}{4}\ln^2\frac{m^2}{M^2}\mathrm{Li}_2\left(1-\sigma\right)+\frac{11\pi^2}{12}\ln\frac{m^2}{M^2}\ln\sigma$$

$$+\frac{1}{4}\ln\frac{m^2}{M^2}\ln^3\sigma+\frac{5}{24}\ln^4\sigma+\frac{1}{2}\left(\ln\frac{m^2}{M^2}+6\ln\sigma\right)S_{1,2}(1-\sigma)$$

$$-\frac{3}{2}\left(\ln\frac{m^2}{M^2}+2\ln\sigma\right)\mathrm{Li}_3\left(1-\sigma\right)+\frac{1}{2}\ln\frac{m^2}{M^2}\ln\sigma\mathrm{Li}_2\left(1-\sigma\right)$$

$$+\frac{1}{2}\ln^2\sigma\mathrm{Li}_2\left(1-\sigma\right)+\frac{7\pi^2}{12}\ln^2\sigma-\zeta_3\ln\sigma+4S_{1,3}(1-\sigma)$$

$$\left.-2S_{2,2}(1-\sigma)+\frac{1}{2}\left[\mathrm{Li}_2\left(1-\sigma\right)\right]^2+\frac{\pi^4}{72}+\mathcal{O}(\varepsilon)\right\}$$

Conclusion

New class of 3-point 2-loop diagrams that can be analytically calculated in terms of the *standard* polylogarithmic functions.

This suggests that other similar contributions (crossed diagrams, etc.) can also be calculated.